

A DIAMETER BOUND FOR CLOSED HYPERBOLIC 3-MANIFOLDS.

MATTHEW E. WHITE

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1. INTRODUCTION

Let $P = \langle a_1 \dots a_n \mid r_1 \dots r_m \rangle$ be a presentation for a group G , then the **length** of P is given by

$$\ell(P) = \sum_{i=1}^m \ell(r_i)$$

where $\ell(r_i)$ is the word length of the relation r_i . In [C], Cooper proves the following:

Theorem 1.0 (Cooper). *For every closed hyperbolic 3-manifold M , the following relationship holds:*

$$\text{vol}(M) < \pi \ell(P)$$

for every presentation P of $\pi_1(M)$.

In this chapter, we are going to prove a similar theorem for the diameter of a closed, orientable hyperbolic 3-manifold:

Theorem 5.9. *There is an explicit constant $0 < R$ such that if M is a closed, connected, hyperbolic 3-manifold, and P is any presentation of its fundamental group, then $\text{diam}(M) < R(\ell(P))$.*

By Margulis' result, in our setting diameter and injectivity radius are inversely related. Thus, our theorem can also be viewed as a lower bound on injectivity radius; that is, with the above hypothesis, $\text{inj}(M) > \frac{1}{R(\ell(P))}$. It is known that that infinitely many closed, hyperbolic 3-manifolds of volume less than a given upper bound may be obtained by hyperbolic Dehn surgery on a finite list of compact manifolds. But only finitely many of these closed manifolds have diameter less than a given upper bound. Thus, our results provide a sharper version of Theorem 1.0.

An outline of the proof is the following: we construct a straight 2-complex from a fundamental group presentation which maps into the manifold π_1 -isomorphically. Assuming the diameter of the manifold is very large compared to the presentation length, Margulis provides us with a deep solid torus surrounding a short geodesic core. It turns out that the 2-complex cannot be homotoped to be disjoint from the solid torus. We consider the subcomplex that maps into the solid torus. From this subcomplex, we construct another 2-complex which simultaneously has a large torsion subgroup in first homology and a small triangulation. This is a contradiction.

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Our discussion is organized as follows: In Section 2, we define the presentation complex and quote some of Cooper's results from his proof of the bound for volume. Section 3 discusses Margulis' Lemma and describes its influence on the geometry of the presentation complex. Section 4 shows that we can construct a 2-complex with first homology bounded by presentation length which maps into the Margulis solid torus. The main theorem is proved in Section 5.

2. THE TRIANGULAR PRESENTATION AND ITS COMPLEX

Given a presentation P of a group G , a **presentation complex** K for P is a 2-complex with fundamental group G constructed as follows. The 1-skeleton $K^{(1)}$ is a wedge of circles, one for each generator of P . There is one 2-cell \mathbb{D} for each relator r of P , where $\partial\mathbb{D}$ is glued to $K^{(1)}$ along a loop representing r . If M is a closed hyperbolic 3-manifold, P is a presentation of $\pi_1(M)$, and K is a presentation complex for P , then there is a map $f : K \rightarrow M$ that induces an isomorphism of fundamental groups.

A presentation P for a group G is called **minimal** if it has minimal length over all presentations of G . If M is a closed hyperbolic 3-manifold, then $\pi_1(M)$ is finitely generated so a minimal presentation for $\pi_1(M)$ exists.

A presentation is called **triangular** if every relation has length 3. We have the following:

Proposition 2.0. *If P is a finite presentation of a group G , then there is a triangular presentation P' of G such that $\ell(P') \leq 3\ell(P)$.*

Sketch Proof Let $P = \langle a_1 \dots a_n \mid r_1 \dots r_m \rangle$. Proceed by induction on $\ell(P)$. Remove all relators of the form $a_i = a_j$ by replacing every occurrence of a_j with a_i . For a relator of the form $a_i^2 = 1$, add a generator b and the relations $a_i^2 b = 1$, $b_i b_i b^{-1} = 1$. If there is a relator of length at least 4 $a_{i_1}^{e_1} a_{i_2}^{e_2} \dots a_{i_k}^{e_k}$, add a generator b and two relations $a_{i_1}^{e_1} a_{i_2}^{e_2} b = 1$, $b^{-1} a_{i_3}^{e_3} \dots a_{i_k}^{e_k} = 1$. \square

Now, let $P = \langle a_1 \dots a_n \mid r_1 \dots r_m \rangle$ be a triangular presentation for $\pi_1(M)$. We may choose $f : K^{(1)} \rightarrow M$ so that the image has straight edges (each edge is a geodesic lasso at $f(v)$). The image of each relator r_i lifts to a loop in \mathbb{H}^3 with three geodesic edges. These edges bound a hyperbolic triangle. Extend f to map the 2-cell corresponding to r_i to this triangle. Repeat this procedure for each relator. It follows that $f : K \rightarrow M$ is a π_1 -isomorphism. We call $f : K \rightarrow M$ a **triangular complex**. Clearly, each relation in P corresponds to a disc in K which lifts to a genuine hyperbolic triangle in \mathbb{H}^3 . This construction is used by Cooper [C] to prove Theorem 1.0. The proof of this theorem also provides these two important propositions:

Proposition 2.1 (Bounded Area). *Using the pull-back metric, $\text{Area}(K) \leq \pi\ell(P)$.*

Proposition 2.2 (Invariant Intersection). *Suppose K is a 2-complex and $f : K \rightarrow M$ induces a π_1 -isomorphism. Then every essential loop in M meets $f(K)$.*

3. GEOMETRIC PRELIMINARIES

In this section, we use the results above together with Margulis' well-known and fundamental result to develop a picture of the "thick-thin" decomposition for

a hyperbolic 3-manifold which is relevant to our setting. A nice (and detailed) overview of this material may be found in [BP]. We are concerned with closed, connected, oriented, hyperbolic 3-manifolds. Therefore, unless specifically noted otherwise, every manifold we consider below shall be of this type. By the ϵ -**thick part** of M , we mean

$$M_{[\epsilon, \infty)} = \{ x \in M \mid \text{inj}(x) \geq \epsilon \}.$$

Likewise, the ϵ -**thin part** is

$$M_{(0, \epsilon]} = \text{closure}(M - M_{[\epsilon, \infty)}).$$

The structure of the thin part of a closed hyperbolic 3-manifold is displayed in the following result due to Margulis [BP]:

Theorem 3.0 (Margulis). *Suppose that M is a closed, orientable hyperbolic 3-manifold. There exists a universal constant $\tilde{\epsilon}$ such that if $\epsilon \leq \tilde{\epsilon}$, then $M_{(0, \epsilon]}$ is a disjoint union of solid tori (“Margulis Tubes”). The degenerate case where one of the solid tori is S^1 may occur. Moreover, the core curve of each solid torus is a geodesic in M which generates an infinite cyclic subgroup of $\pi_1(M)$.*

We can now use Theorem 1.0 to obtain the analog of the main theorem for the thick part of a hyperbolic manifold:

Lemma 3.1 (Thick Diameter is Bounded). *There is a universal constant C_1 such that given a closed hyperbolic 3-manifold M with $\text{diam}(M_{[\tilde{\epsilon}, \infty)}) \leq C_1 \text{vol}(M)$ for every presentation P of $\pi_1(M)$. Thus, $\text{diam}(M_{[\tilde{\epsilon}, \infty)}) \leq C_1 \ell(P)$ for every presentation P of $\pi_1(M)$.*

Proof Since $M_{[\tilde{\epsilon}, \infty)}$ is compact, there exist points x and y in $M_{[\tilde{\epsilon}, \infty)}$ such that $d(x, y) = \text{diam}(M_{[\tilde{\epsilon}, \infty)})$. Now, $M_{[\tilde{\epsilon}, \infty)}$ is path connected, so there is a path L in $M_{[\tilde{\epsilon}, \infty)}$ with endpoints x and y . By considering $M_{[\frac{\tilde{\epsilon}}{2}, \infty)}$ if necessary, we can arrange that each point on L is contained in a ball of radius $\frac{\tilde{\epsilon}}{4}$ that is isometric to a standard ball $B(\frac{\tilde{\epsilon}}{4})$ in \mathbb{H}^3 .

There is a covering of L so that each ball B meets at most two others in the covering and every such intersection is a point of tangency between the boundary of B and the boundary of another ball in the covering. It follows that since the interiors of the balls do not intersect, the volume of the covering is the sum of the volumes of the balls in the covering. Our construction implies that at most $\frac{\text{vol}(M)}{B(\frac{\tilde{\epsilon}}{4})} + 2$ balls cover L . Then using Theorem 1.0,

$$\text{diam}(M_{[\epsilon, \infty)}) \leq 2\tilde{\epsilon} \left(\frac{\text{vol}(M)}{B(\frac{\tilde{\epsilon}}{4})} + 2 \right) \leq \frac{2\tilde{\epsilon}(\pi + 2)}{B(\frac{\tilde{\epsilon}}{4})} \ell(P)$$

for every presentation P of $\pi_1(M)$. Therefore, put $C_1 = \frac{2\tilde{\epsilon}(\pi + 2)}{B(\frac{\tilde{\epsilon}}{4})}$ and the proof is complete. \square

Notice that Proposition 2.2 (invariant intersection) implies that if $f : K \rightarrow M$ induces an isomorphism of fundamental groups, then $f(K)$ meets the core curve of every Margulis Tube in M . Our approach shall be to study the intersection of

$f(K)$ with a particular tube, the idea being that the geometry of a solid torus is simple enough to impose significant constraints on the map f . One can combine Theorem 1.0 and Lemma 3.1 in the following way: given M , if $\text{diam}(M)$ is very large compared to $\ell(\pi_1(M))$, then $\text{diam}(M)$ is very large compared to $\text{vol}(M)$. Both $\text{diam}(M_{[\tilde{\epsilon}, \infty)})$ and $\text{vol}(M)$ are bounded above by a constant multiple of $\ell(\pi_1(M))$. Hence, one of the Margulis Tube components of $M_{(0, \tilde{\epsilon}]}$ must have large diameter. This is the content of the following:

Proposition 3.2 (Deep Tube). *Let M be as above, let $C > C_1 + 1$ and suppose that $\text{diam}(M) \geq C\ell(P)$ for some presentation P of $\pi_1(M)$. Then there exists a Margulis Tube $V_{\tilde{\epsilon}} \subset M$ such that $\text{diam}(V_{\tilde{\epsilon}}) \geq \frac{1}{2}(\text{diam}(M) - C_1\text{vol}(M))$. Thus, $\text{diam}(V_{\tilde{\epsilon}}) \geq \frac{1}{2}(C - 1)\ell(P)$.*

Proof Given M , suppose that $\text{diam}(M) \geq C\ell(P)$ for some presentation P of $\pi_1(M)$. By compactness, there exist x and y in M so that $d(x, y) \geq C\ell(P)$. Lemma 3.1 (Thick Diameter is Bounded) shows that at least one of the points is contained in $M_{(0, \tilde{\epsilon}]}$. Thus, if one of the points, say y , is contained in the thick part of M , and $x \in V_{\tilde{\epsilon}} \subset M_{(0, \tilde{\epsilon}]}$, then $d(y, \partial V_{\tilde{\epsilon}}) \leq C_1\ell(P)$. This implies that

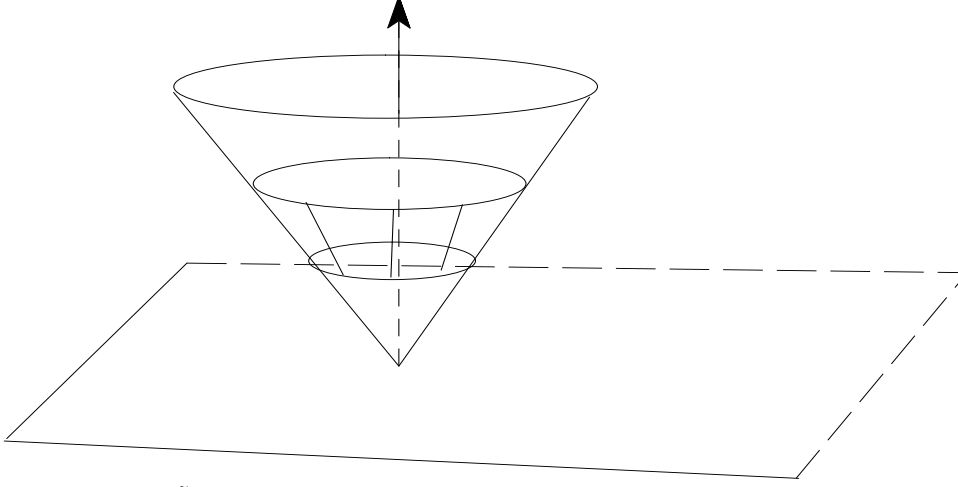
$$(C - C_1)\ell(P) \leq d(x, y) - d(y, \partial V_{\tilde{\epsilon}}) \leq d(x, V_{\tilde{\epsilon}})$$

whence $\text{diam}(V_{\tilde{\epsilon}}) \geq (C - 1)\ell(P)$. A similar approach shows that if both x and y are contained in $M_{(0, \tilde{\epsilon}]}$, then:

$$\text{diam}(V_{\tilde{\epsilon}}) \geq \frac{1}{2}(C - 1)\ell(P).$$

□

We shall use the notation $V_{\tilde{\epsilon}}$ for a deep tube of M provided by the above proposition. It is useful for us to have a picture of the geometry of $\partial V_{\tilde{\epsilon}}$. To develop this, consider the upper-half space model of \mathbb{H}^3 . Let γ denote the core curve of $V_{\tilde{\epsilon}}$. In $\pi_1(M)$, $[\gamma]$ corresponds to a loxodromic isometry $\tilde{\gamma}$ of \mathbb{H}^3 . We may assume that the axis of $\tilde{\gamma}$ is the z -axis, so that a component of the preimage of $V_{\tilde{\epsilon}}$ under the universal cover $\rho : \mathbb{H}^3 \rightarrow M$, denoted $\tilde{V}_{\tilde{\epsilon}}$, is a neighborhood of the z -axis. This appears as an infinite Euclidean cone with (ideal) vertex at the origin. The isometry $\tilde{\gamma}$ acts by translation along and rotation around the z -axis, so that evidently a fundamental domain for $V_{\tilde{\epsilon}}$ is a horizontal “pancake,” that is, in cylindrical coordinates the fundamental domain is a set of the form $\{(r, \theta, z) \mid a \leq z \leq b, \frac{r}{z} \leq c\}$. The covering translation glues this set top to bottom with a twist. If $C > C_1$ and $\text{diam}(M) \geq C\ell(P)$, Margulis and Proposition 3.2 (deep tube) tell us that the radius of this “pancake” is at least $\frac{1}{2}C\ell(P) - \tilde{\epsilon}$. Clearly, $\partial \tilde{V}_{\tilde{\epsilon}}$ is an annulus and the boundary components of this annulus are lifts of the meridian of $V_{\tilde{\epsilon}}$. These facts, coupled with the results in [C], give us very nice bounds on the geometry of $\partial V_{\tilde{\epsilon}}$.

Figure 3.0. \tilde{V}_ϵ in the upper-half space model of \mathbb{H}^3

Lemma 3.3 (Boundary Torus). *Let M be a closed hyperbolic 3-manifold. Let P be any presentation for $\pi_1(M)$. Suppose $\text{diam}(M) \geq C\ell(P)$ where $C > C_1 + 1$. Let V_ϵ is a deep tube in M given by Proposition 3.2. The following statements then hold for ∂V_ϵ :*

- (i) *The induced metric on ∂V_ϵ is Euclidean.*
- (ii) *For every $x \in \partial V_\epsilon$, we have $\text{inj}(x) \geq \epsilon$.*
- (iii) *A loop α on ∂V_ϵ homologous to a power of the meridian has $\text{length}(\alpha) \geq 2\pi \sinh(\frac{1}{2}C\ell(P))$.*

Proof Certainly ∂V_ϵ is topologically a torus. Thus, (i) will be established if we show that for each point x in ∂V_ϵ , the curvature at x is 0. We note that ∂V_ϵ is homogeneous. To see this, let x and y be distinct points in ∂V_ϵ . Choose lifts \tilde{x} and \tilde{y} of these points to $\tilde{\partial V}_\epsilon$. There is a rotation and translation about the z -axis in \mathbb{H}^3 taking \tilde{x} to \tilde{y} . Since this commutes with the covering isometry $\tilde{\gamma}$, it gives a well-defined isometry of ∂V_ϵ which takes x to y . So it suffices to prove that the curvature is 0 at one point in ∂V_ϵ . Clearly, the curvature of ∂V_ϵ is not positive since otherwise the Gauss-Bonnet Theorem implies that ∂V_ϵ is a sphere. On the other hand, since $\rho : \mathbb{H}^3 \rightarrow M$ is a local isometry, the curvature at a point \tilde{x} in $\tilde{\partial V}_\epsilon$ must equal the curvature at $x = \rho(\tilde{x})$ in ∂V_ϵ . But $\tilde{\partial V}_\epsilon$ is cylinder with many distinct simple closed geodesics; in fact, every horosphere centered at ∞ intersects $\tilde{\partial V}_\epsilon$ in a simple closed geodesic. A second application of Gauss-Bonnet shows this fact implies the curvature of $\tilde{\partial V}_\epsilon$ cannot be negative. This completes the proof since the cylinder itself is homogeneous. Statement (ii) follows at once from our previous discussion. For (iii), suppose α is a power of the meridian. Since the metric on ∂V_ϵ is Euclidean, this loop is at least as long as a geodesic representative of the meridian. A geodesic meridian lifts to a “horizontal” circle; that is, the intersection of a horosphere centered at ∞ with $\tilde{\partial V}_\epsilon$. This has length $2\pi \sinh(\frac{1}{2}C\ell(P))$. \square

Proposition 3.4. *Let $B(\frac{\epsilon}{2})$ denote the standard ball of radius $\frac{\epsilon}{2}$ in \mathbb{H}^3 . There is a constant $C_2 > C_1 + 1$ such that if M is a closed hyperbolic 3-manifold, $\text{diam}(M) \geq C_2\ell(P)$, and V_ϵ is a deep tube in M , then $\frac{1}{2}\text{Vol}(B(\frac{\epsilon}{2})) \leq \text{Area}(\partial V_\epsilon) \leq \text{Vol}(M)$. Thus, $\text{Area}(\partial V_\epsilon) \leq 2\pi\ell(P)$.*

Proof A simple calculation shows that for a hyperbolic solid torus of radius L ,

$$\frac{\text{Area}(\partial V_{\tilde{\epsilon}})}{\text{Vol}(V_{\tilde{\epsilon}})} = 2 \frac{\cosh(L)}{\sinh(L)}$$

Thus, it suffices to find bounds on $\text{Vol}(V_{\tilde{\epsilon}})$. Choosing a point x on $\partial \tilde{V}_{\tilde{\epsilon}}$, we have that $\text{inj}(x) = \tilde{\epsilon}$. Hence, the restriction of the universal covering map $\rho : \mathbb{H}^3 \rightarrow M$ to $B(x, \frac{\tilde{\epsilon}}{2})$ is an embedding. If $C_2 > C_1 + 1$ is chosen to be sufficiently large, then the tube radius is very large. Hence, in our setting $B(x, \frac{\tilde{\epsilon}}{2}) \cap \partial V_{\tilde{\epsilon}}$ is approximately a hemisphere centered at x . This means that for a small δ' which depends only on C_2 ,

$$\text{Vol}(V_{\tilde{\epsilon}}) \geq \frac{1}{2} \text{Vol}(B(x, \frac{\tilde{\epsilon}}{2})) - \delta'$$

where $\delta' \rightarrow 0$ as $C_2 \rightarrow \infty$. The above shows that we may assume

$$\frac{1}{2} \text{Vol}(B(x, \frac{\tilde{\epsilon}}{2})) - \delta' \geq \frac{1}{4} \text{Vol}(B(x, \frac{\tilde{\epsilon}}{2}))$$

whence

$$\text{Area}(\partial V_{\tilde{\epsilon}}) \geq 2(\frac{1}{2} \text{Vol}(B(x, \frac{\tilde{\epsilon}}{2})) - \delta') + \delta \geq 2(\frac{1}{4} \text{Vol}(B(x, \frac{\tilde{\epsilon}}{2}))).$$

The right hand inequality of then follows since

$$\text{Area}(\partial V_{\tilde{\epsilon}}) \leq 2\text{vol}(V_{\tilde{\epsilon}}) < 2\pi\ell(P).$$

□

Now suppose X and Y are closed loops on a Euclidean torus. We write $\Delta(X, Y)$ for the algebraic intersection number of X and Y . Note that for a pair closed geodesics on a torus, the algebraic and geometric intersection numbers are equal. We prove a simple result about intersections of closed loops on a Euclidean torus with "bounded geometry" that is quite useful in our setting.

Proposition 3.5. *Suppose T is a Euclidean Torus with area A and injectivity radius R . There is a basis $\{[X], [Y]\}$ for $\pi_1(T)$ with X and Y closed geodesics such that:*

- (i) $\Delta([X], [Y]) = 1$,
- (ii) $\ell(X) = R$,
- (iii) $\ell(Y) \leq \frac{2A}{\sqrt{3}R}$.

Proof Fix a universal cover $\rho : \mathbb{R}^2 \rightarrow T$ and choose a point x on T . Since $\text{inj}(T) = R$, there is a closed geodesic loop X based at x with length R . The lifts of X thus form a set of parallel lines in the plane, which for simplicity we may assume are horizontal. For each lift \tilde{X} , the set $\tilde{X} \cap \rho^{-1}(x)$ is a collection of points such that each adjacent pair is separated by a distance of R . A straightforward trigonometry calculation shows that each adjacent pair of lifts, say \tilde{X} and \tilde{L} , must have the property that $d(\tilde{X}, \tilde{L})$ is at least $\frac{\sqrt{3}}{2}\epsilon$ as otherwise we can join a pair of points in $\rho^{-1}(x)$ by a path of length less than R . This would produce an essential loop in T of length less than R which is impossible. On the other hand, it is easy

to construct a fundamental domain for the action of $\pi_1(T)$ on \mathbb{R}^2 as follows: choose a point \tilde{x}_1 on $\tilde{X} \cap \rho^{-1}(x)$. There is a point \tilde{y} on $\tilde{L} \cap \rho^{-1}(x)$ with the property that

$$d(\tilde{x}_1, \tilde{y}) = \min\{d(\tilde{x}_1, z) \mid z \in \rho^{-1}(x) - \tilde{x}_1\}.$$

By joining \tilde{x}_1 and \tilde{y} with a straight segment and then constructing a parallel segment between adjacent points on \tilde{X} and \tilde{L} respectively, we obtain a parallelogram which is a fundamental domain for $\pi_1(T)$. Also, the segment joining x_1 and y projects to an essential loop Y on T which by construction has $\Delta([X], [Y]) = 1$. By a short calculation, we conclude that $\ell(Y) \leq \frac{2A}{\sqrt{3}R}$, since the area of this parallelogram is exactly A . \square

Definition 3.6. *Suppose T is a Euclidean Torus with area A and injectivity radius R . A basis $\{[X], [Y]\}$ for $\pi_1(T)$ that satisfies the conditions of Proposition 3.5 is a **short basis** for $\pi_1(T)$.*

Proposition 3.7. *Suppose T is a Euclidean Torus with injectivity radius R . Let L be an essential loop on T . If $\{[X], [Y]\}$ is a short basis for $\pi_1(T)$, then $[L] = a[X] + b[Y]$ where $\max\{|a|, |b|\} \leq \frac{2 \text{length}(L)}{\sqrt{3}R}$.*

Proof We can tile the plane with copies of the parallelogram fundamental domain for T as constructed in the proof of Proposition 3.5. Since the distance between parallel sides of a given parallelogram is at least $\frac{\sqrt{3}}{2}S$, the lift of a geodesic representative of $[L]$ hits at most $\frac{2 \text{length}(L)}{\sqrt{3}R}$ parallelograms. In particular, $\Delta([L], [X])$ and $\Delta([L], [Y])$ are both bounded above by $\frac{2 \text{length}(L)}{\sqrt{3}R}$. \square

We can apply the above results to our setting at once since the injectivity radius on $\partial V_{\tilde{\epsilon}}$ is approximately the Margulis constant $\tilde{\epsilon}$. In fact the injectivity radius on the boundary torus is slightly larger. To see this, lift a short essential loop on $\partial V_{\tilde{\epsilon}}$ to \mathbb{H}^3 . This gives a path between distinct points on $\partial V_{\tilde{\epsilon}}$. The geodesic path in \mathbb{H}^3 joining these points has length at least $2\tilde{\epsilon}$, so that the lifted path is slightly larger.

Corollary 3.8. *Let V denote a margulis tube of M . There exists a short basis $\{[X], [Y]\}$ for $\pi_1(\partial V_{\tilde{\epsilon}})$ such that if L is any essential loop on ∂V , then $[L] = a[X] + b[Y]$ where $\max\{|a|, |b|\} \leq \frac{2 \text{length}(L)}{\sqrt{3}\tilde{\epsilon}}$.*

Suppose that $f : K \rightarrow M$ is a triangular complex. Lemma 2.2 (Invariant Intersection) shows that $f(K)$ must meet V . This gives us two important subsets of K . Since $f(K)$ must meet the core of $V_{\tilde{\epsilon}}$, the set $f^{-1}(V_{\tilde{\epsilon}})$ is mapped very far into the deep tube. Define the **pull-back boundary** of $f^{-1}(V_{\tilde{\epsilon}})$ to be the set $f^{-1}(\partial V_{\tilde{\epsilon}})$. It follows that the pull-back boundary is a subset of the topological boundary of $f^{-1}(V_{\tilde{\epsilon}})$. However, we note that in general the pull-back boundary does not equal the topological boundary. Lemma 3.3 (Boundary Torus) allows us to completely describe these sets geometrically. We obtain three types of intersection set as shown in Figure 3.1. We call these intersection sets **0-handles**, **1-handles**, and **monkey-handles** respectively. Notice that a *monkey-handle* appears as a disc with six edges, three of which join adjacent edges of the triangle. A given triangle can contain at most one *monkey-handle*.

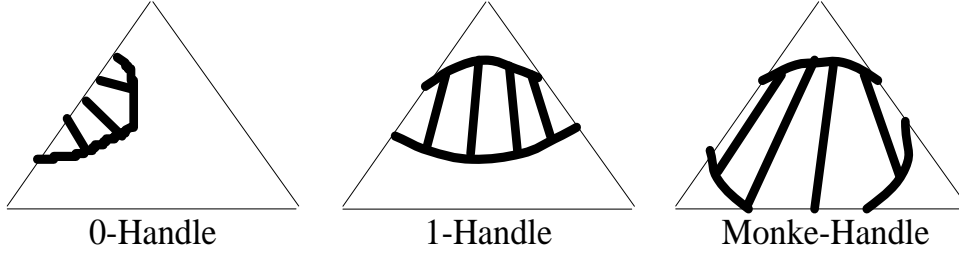


Figure 3.1.

Lemma 3.9. *Let $f : K \longrightarrow M$ be a triangular complex. Let V be any Margulis tube in M . Then:*

- (i) $f^{-1}(V_\epsilon)$ is a finite union of 0-handles, 1-handles, and monkey-handles, together with a collection of discs contained in the interiors of triangles of K .
- (ii) The pull-back boundary is a (possibly not connected) graph.
- (iii) The number of monkey-handles in $f^{-1}(V_\epsilon)$ is at most the number of triangles in K .
- (iv) Removal of the interiors of the monkey-handles from $f^{-1}(V_\epsilon)$ yields an I-bundle over a graph.

Proof We may assume that the single vertex of K is mapped by f into the thick part of M . Since every relation in $\pi_1(M)$ corresponds to a disc \mathbb{D} in K which lifts to a hyperbolic triangle in \mathbb{H}^3 , we may view \mathbb{D} as a hyperbolic triangle with vertices identified to a single point. In particular, we can understand K_v by looking at the intersection of each lifted triangle in \mathbb{H}^3 with $\partial\tilde{V}_\epsilon$. By an arbitrarily small adjustment, we can assume that the intersection of a hyperbolic triangle with the conical boundary, $\partial\tilde{V}_\epsilon$, is transverse. Both \tilde{V}_ϵ and the triangle are convex. This implies their intersection is convex, so the intersection must be a disc. Also, each triangle edge is convex, so the intersection of each edge with this disc is either empty or an interval. We readily obtain the three types pictured in Figure 3.1. To see that the union is finite, consider a fundamental domain for V_ϵ . A fundamental domain for K in \mathbb{H}^3 is a finite union of (compact) hyperbolic triangles. The action of $\pi_1(M)$ on \mathbb{H}^3 is properly discontinuous. Therefore, only finitely many translates under $\pi_1(M)$ of a given triangle in K hit the fundamental domain for V_ϵ . This proves (i) and (ii). Part (iii) is obvious. Also, (iv) follows since the complement of the interiors of the *monkey*-handles in K_v is built by glueing 0-handles and 1-handles (which are I-bundles over an interval) along interval fibres. \square .

Remark In our subsequent arguments, we do not need to consider discs in $f^{-1}(V_\epsilon)$ which are contained in the interior of a triangle of K . Thus, we let $K_V = f^{-1}(V_\epsilon) - \{\text{interior discs}\}$ and define $\partial_f K_V$ to be the corresponding pull-back boundary.

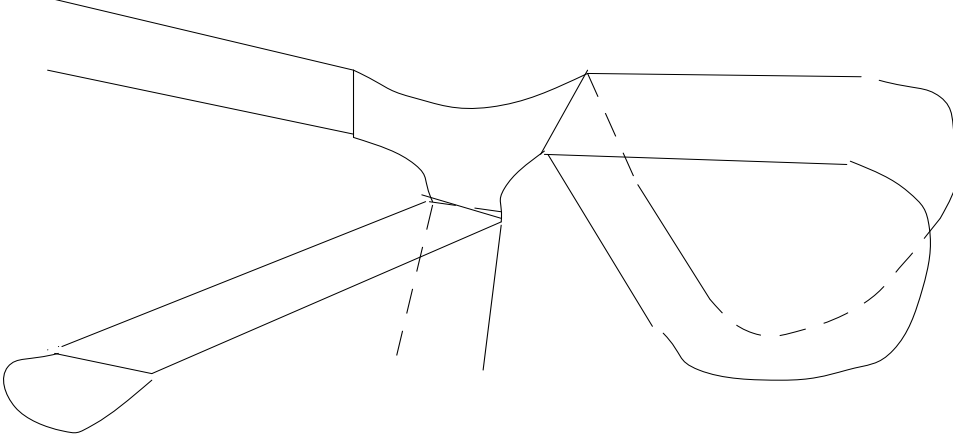


Figure 3.2. A piece of K_V . Note that removing the monkey-handle interiors gives an I-Bundle over a graph.

We use Proposition 2.1 (Bounded Area) to show that K_V has bounded geometry. To explain this, let $\{\partial V_s\}_{s=0}^1$ be a parameterization by distance s of the parallel concentric tori in $\partial V_{\bar{\epsilon}}$; that is, ∂V_s is the parallel torus of distance s from $\partial V_{\bar{\epsilon}}$. Let $\Gamma(s) = K_v \cap f^{-1}(\partial V_s)$. We can show that there is an s such that $0 < s < 1$ and $\text{length}(\Gamma(s))$ is bounded by presentation length. Thus, by replacing $V_{\bar{\epsilon}}$ with V_s if necessary, we may assume that the length of $\partial_f K_v$ is also bounded in terms of presentation length. In section 4, we shall be able to prove results bounding the homology of K_v using this fact.

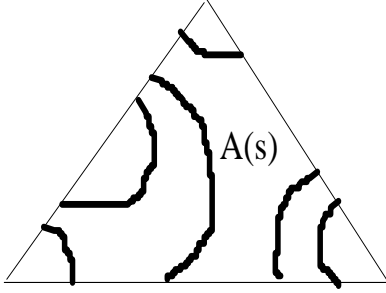


Figure 3.3.

Lemma 3.10 (Short Boundary). *With the notation as above, there is a universal constant $C_3 > C_2$ such that $\text{diam}(M) > C_3 \ell(P)$ implies there exists s_0 with $0 < s_0 < 1$ and $\text{length}(\Gamma(s_0)) < \text{area}(K)$. Thus, $\text{length}(\Gamma(s_0)) < 2\pi \ell(P)$.*

Proof It is sufficient to show this for a single triangle in K . So let T be a triangle in K and consider $A(s) = \Gamma(s) \cap T$. Then $A(s)$ is a collection of embedded arcs in T . The endpoints of each arc are contained in edges of T . At most three of these are tangent to an edge of T ; all others are properly embedded. It follows that:

$$\int_0^1 \text{length}(A(s)) ds \leq \text{Area}(T).$$

Since $\text{Area}(K)$ is the sum of the areas of the triangles in K , the lemma follows at once. \square

Given this result, it is best to introduce new notation: let V denote the Margulis Tube in V_ε with boundary $\partial V = \partial V_{s_0}$, where s_0 is provided by Lemma 3.5 (Short Boundary). It is clear from the above proof that we can choose s_0 so that V is properly contained in V_ε . This assumption simplifies later exposition since $V_\varepsilon - V$ is homeomorphic to the product of a torus and an interval. We shall now set $K_V = f^{-1}(V) - \{\text{interior discs}\}$ so that $\partial_f K_V < 2\pi\ell(P)$.

Corollary 3.11. *Let M be a closed hyperbolic 3-manifold, P a triangular presentation for $\pi_1(M)$ and $f : K \rightarrow M$ a corresponding triangular complex. Assume that $L \subset \partial_f K_V$ is an embedded loop with $f_*([L]) \neq 1$ in $\pi_1(\partial V)$. Then $\text{diam}(M) > C_3\ell(P)$ implies $f_*([L])$ is not a power of the meridian of V .*

Proof By conclusion (iii) of Lemma 3.3 (Boundary Torus), if $f_*([L])$ is a power of the meridian, then we have $\text{length}(f(L)) \geq 2\pi \sinh(\frac{1}{2}C_3\ell(P))$. By our construction, $C_3 > 1$, so certainly $2\pi\ell(P) < 2\pi \sinh(\frac{1}{2}C_3\ell(P))$. But this implies $\text{length}(L) = \text{length}(f(L)) \geq 2\pi\ell(P)$ which contradicts Lemma 3.10 (Short Boundary). \square

Corollary 3.12. *A simple loop in $\partial_f K_V$ is inessential in K if and only if its image under f is inessential in ∂V .*

Remark By the above results, K does not contain a disc which is mapped to a meridian of a deep tube (since this gives enormous area). Also, the image of K intersects every essential loop in V . However, as the following example shows, these facts do not yield a proof. Using the short basis for $\pi_1(\partial V)$ provided by Corollary 3.8, sweep out a 2-complex which hits the core of V essentially by concentrically shrinking the loops X and Y to the core of V . By construction, this complex has boundary length $\text{length}(X) + \text{length}(Y)$ which by Propositions 3.31 and 3.32 is at most $\tilde{\varepsilon} + \frac{4\pi\ell(P)}{\sqrt{3}\tilde{\varepsilon}}$. This means a meridian is too long to be contained in the boundary of the complex. On the other hand, this complex has bounded area and intersects every essential loop in V . The next two sections deal with this complication.

4. BOUNDS ON FIRST HOMOLOGY

In this section, we use the results of Section 3 to establish a bound on the “complexity” of K_V . This amounts to showing that $\text{Rank}(H_1(K_V))$ is small compared to diameter.

Lemma 4.0. *Let G_1, \dots, G_r denote the components of $\partial_f K_V$. There exists a component G_i such that the restriction $f|_{G_i} : G_i \rightarrow \partial V$ has nontrivial induced homomorphism $f|_{G_{1*}} : \pi_1(G_i) \rightarrow \pi_1(\partial V)$.*

Proof If every component of $\partial_f K_V$ has trivial induced homomorphism into $\pi_1(\partial V)$, we may homotop f on each component G_i so that the image of G_i under f is a finite collection of points in ∂V . This contradicts Proposition 2.2 since there is a loop in ∂V that is essential in M and misses each of these points. \square

Lemma 4.1. *Suppose $f : K \rightarrow M$, K is a 2-complex, and f is a π_1 -isomorphism. Let \mathbb{D} be an embedded disc in K with the property that $\partial\mathbb{D} \simeq *$ in $K - \text{int}(\mathbb{D})$ and $\text{closure}(K - \mathbb{D}) \cap \mathbb{D} = \partial\mathbb{D}$. Then the restriction $f|_{K - \text{int}(\mathbb{D})} : K - \text{int}(\mathbb{D}) \rightarrow M$ is a π_1 -isomorphism.*

Proof Let $i : K - \text{int}(\mathbb{D}) \rightarrow K$ and $j : \partial\mathbb{D} \rightarrow K$ denote the inclusion mappings. Let x be a basepoint in $K - \text{int}(\mathbb{D})$ and let $N(j_*(\pi_1(\partial\mathbb{D}, x)))$ denote the

smallest normal subgroup of $\pi_1(K, x)$ containing $j_*(\pi_1(\partial\mathbb{D}, x))$. Van Kampen's Theorem gives that $i_* : \pi_1(K - \text{int}(\mathbb{D}), x) \rightarrow \pi_1(K, x)$ is an epimorphism with kernel $N(j_*(\pi_1(\partial\mathbb{D}, x)))$. The hypothesis gives that $N(j_*(\pi_1(\partial\mathbb{D}, x))) = 1$, so that i_* is an isomorphism. This completes the proof since f_* is an isomorphism and $f|_{K - \text{int}(\mathbb{D})_*} = f_* i_*$ \square

Lemma 4.2. *Let K be a 2-complex, M a 3-manifold, and $f : K \rightarrow M$ a π_1 isomorphism. Suppose γ is an inessential loop in K and form the complex $K' = K \cup_\gamma \mathbb{D}$. There is a map $f' : K' \rightarrow M$ that is a π_1 isomorphism.*

Proof Since γ is inessential, Van Kampen's theorem gives that $\pi_1(K') \cong \pi_1(K)$. Also, $f(\gamma)$ is inessential in M , so there is a map $h : \mathbb{D} \rightarrow M$ with $h(\partial\mathbb{D}) = f(\gamma)$. Extend f over \mathbb{D} using h and the proof is complete. \square

Lemma 4.3. *Suppose $f : \mathbb{D} \rightarrow \partial V$. Then there is a map $g : \mathbb{D} \rightarrow V_\epsilon$ such that the following conditions hold:*

- (i) $g = f$ on $\partial\mathbb{D}$
- (ii) $g \sim f$ rel $\partial\mathbb{D}$
- (iii) $g(\text{int}(\mathbb{D})) \subset V_\epsilon - V$.

Proof Push the interior of $f(\mathbb{D})$ out into the collar of ∂V contained in $V_\epsilon - V$. \square

We can actually make many such maps with that property that any two maps send the interior of the disc to disjoint sets in $V_\epsilon - V$. We obtain the following useful corollary as a consequence:

Corollary 4.4. *Suppose γ is an inessential loop in $\partial_f K_V$. Let $K' = K \cup_\gamma \mathbb{D}$. Then there is a π_1 isomorphic map $f' : K' \rightarrow M$ such that $K'_V = K_V$.*

Proof Let $g : \mathbb{D} \rightarrow V_\epsilon$ denote the map provided by Lemma 4.3. Then the map $f' = f \cup g$ satisfies our requirements. \square

Let K_I denote $\text{closure}(K_V - \{\text{interior(monkey-handles)}\})$. We also define $\mathcal{N}(K_I)$ by taking K_I together with a small product neighborhood of its boundary in K . We note in passing that this definition ensures that the pull-back boundary of this neighborhood $\partial_f \mathcal{N}(K_I)$ maps into $V_\epsilon - V$. By Lemma 3.9, K_I is an I-bundle over a graph. We wish to show that we can modify K so that $\text{rank}(H_1(K_I))$ is bounded by $\ell(P)$.

To do this, we are going to define a surgery procedure made possible by the above two lemmas. We are motivated by two of our constraints: $\text{length}(\partial_f K_V)$ is bounded above and $\text{inj}(\partial_f K_V)$ is bounded below. Thus, if the rank of $H_1(K_I)$ is large, then there are many short inessential loops in $\partial_f K_V$. The point of our procedure will be to “snap” all of the annuli and mobius bands in K_I that are inessential in K by removing a single 1-handle from each. To preserve the π_1 isomorphism, we must make sure that the resulting new loop remains inessential in the new complex $K - \{1\text{-handle}\}$. We treat the annulus case first. We can attach a pair of discs to the boundary of the original inessential annulus in $\mathcal{N}(K_I)$. By Lemma 4.3, we can ensure that the interiors of these discs miss V , so that K_V remains unchanged; that is, we glue a disc onto each boundary component to produce a 2-sphere. Removal of the 1-handle thus gives an inessential loop. The effect is that K_I appears to lose a 1-handle.

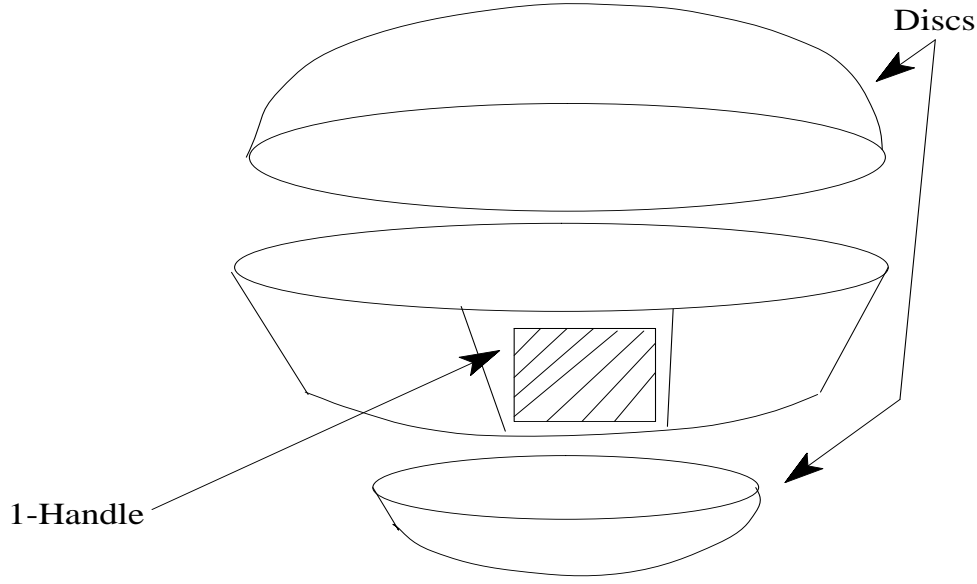


Figure 4.0.

Lemma 4.5 (Inessential Annulus). *Suppose A is an embedded annulus in K_I with $\partial A \subset \partial_f K_I$. Assume that $\partial A \simeq *$ in K . Let H denote any one of the 1-handles contained in A . Define H^+ to be the union of H and a small neighborhood of $\partial_f K_I \cap H$ in the triangle of K containing H . There is a 2-complex K' with $K - \text{int}(H^+) \subset K'$ and a map $f' : K' - \text{int}(H^+) \rightarrow M$ is a π_1 isomorphism. Moreover, $K'_V = K_V - H$.*

Proof By Corollary 3.12, both components of ∂A map to loops that are inessential in ∂V . Using Corollary 4.4 twice, we may attach a disc to each boundary component of $A \cup H^+$ to build the π_1 isomorphic 2-complex $f' : K' \rightarrow M$. The union of these two discs with $A \cup H^+$ is an embedded S^2 in K' . Therefore, ∂H^+ is inessential in $K' - \text{int}(H^+)$. By Lemma 4.1, $f' : K' - \text{int}(H^+) \rightarrow M$ is a π_1 isomorphism. It is then obvious that $K'_I = K_I - H$. \square

Remark Since we define the pull-back boundary as $\partial_f K_V = f^{-1}(\partial V)$, application of Lemma 4.5 (Inessential Annulus) does not increase the length of $\partial_f K_V$.

The mobius band case is more complicated, but can be approached in a similar way. We say that an embedded mobius band B in K_V with $\partial B \subset \partial_f K_V$ is **inessential** if ∂B is inessential in K . This terminology is justified by the following proposition:

Proposition 4.6. *Suppose B is an inessential mobius band in K_V . Let γ denote the core curve of B . Then $\gamma \simeq *$ in K .*

Proof Since γ^2 is homotopic to ∂B , $[\gamma]^2 = 0$ in $\pi_1(M)$. Since M is a closed hyperbolic 3-manifold, $\pi_1 M$ is torsion free. The map $f : K \rightarrow M$ is a π_1 -isomorphism, so γ is contractible in K . \square

Suppose B is an inessential mobius band in K_I . We glue two discs onto B to obtain $f : K' \rightarrow M$ such that the first disc contributes a 0-handle to K'_I and the second disc is mapped into $V_\varepsilon - V$ (and thus is disjoint from K'_I). This is done in such a

way that removing a 1-handle from B in K' does not change $\pi_1 K'$. Since we plan to repeat this procedure for each inessential mobius band, we must ensure that the new attached 0-handle does not increase $\text{length}(\partial_f K_I)$ too much. Let B denote an inessential mobius band in $\mathcal{N}(K_I)$. Thus, every closed loop in B is null homotopic in K .

Definition 4.7. *A good core γ for B is an embedded loop such that the following conditions hold:*

- (i) $[\gamma]$ generates $\pi_1(B)$,
- (ii) γ is the union of a pair of arcs denoted $\mathcal{A} \cup \mathcal{L}$ where \mathcal{A} is an arc on $\partial \mathcal{N}(K_I)$, \mathcal{L} is a properly embedded geodesic arc in B , and $\mathcal{A} \cap \mathcal{L}$ is a pair of distinct points (see Figure 4.1),
- (iii) $f(\gamma)$ is embedded in M .

We use condition (iii) since we will attach a disc to the good core: we must control the resulting intersection of the image of this disc with our deep tube.

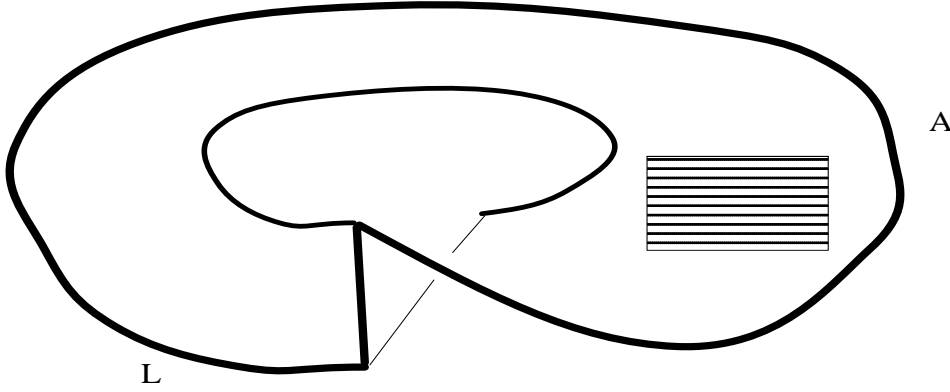


Figure 4.1.

Since $f(\gamma) \simeq *$, there exists a lift $\tilde{\gamma}$ to \mathbb{H}^3 which is an embedding into \tilde{V}_ϵ . Notice that \tilde{V}_ϵ and \tilde{V} are a pair of concentric cones about the z -axis in \mathbb{H}^3 . \mathcal{L} lifts to an arc $\tilde{\mathcal{L}}$ which is properly embedded in V while \mathcal{A} lifts to an arc which is embedded in a concentric cone about the z -axis in $\tilde{V}_\epsilon - \text{int}(V)$. The image $f(\tilde{\gamma})$ thus bounds an embedded disc in \mathbb{H}^3 which intersects V in a single 0-handle. We construct this disc as follows: $\tilde{\mathcal{A}}$ intersects $\partial \tilde{V}$ in a pair of points. Choose the shortest path \mathcal{P} on $\partial \tilde{V}$ joining these two points. Notice that by construction $\text{length}(\mathcal{P}) < \text{length}(\tilde{\mathcal{A}}) < \text{length}(\partial B)$.

Now $\tilde{\mathcal{A}} \cup \mathcal{P}$ is a circle which bounds an embedded disc \mathbb{D}_1 . Likewise, $\mathcal{P} \cup \mathcal{L}$ bounds an embedded disc \mathbb{D}_2 . Put $\mathbb{D} = \mathbb{D}_1 \cup \mathbb{D}_2$. Then, by Lemma 4.2, we may attach \mathbb{D} to γ to obtain a 2-complex $f' : K' \rightarrow M$ with f'_* a π_1 isomorphism and $K'_I = K_I \cup \{0\text{-handle}\}$. Furthermore, $\text{length}(\partial_f K'_I) < \text{length}(\partial_f K_I) + \text{length}(B)$. Our surgery argument is then completed by the next proposition.

Proposition 4.8. *Suppose B is a mobius band. Let \mathbb{D}_X be an embedded disc in B with boundary X . Assume that $\text{interior}(\mathbb{D}_X)$ is disjoint from ∂B and from a good core γ . Form the space \mathcal{X} by attaching one disc \mathbb{D}_γ to the core curve and another disc $\mathbb{D}_{\partial B}$ to the boundary of the mobius band. Then $\mathcal{X} - \text{interior}(\mathbb{D}_X)$ is simply connected.*

Proof There is a retraction of $B - \text{interior}(\mathbb{D}_X)$ onto a closed trivalent graph G which has fundamental group $\mathbb{Z} * \mathbb{Z}$. This retraction maps ∂B and γ to a pair of generators for $\pi_1(G)$. Therefore, the space $\mathcal{X} - \text{interior}(\mathbb{D}_X)$ is homeomorphic to G with a pair of discs glued onto a generating set for $\pi_1(G)$ so that $\pi_1(\mathcal{X} - \text{interior}(\mathbb{D}_X))$ is trivial. \square

Lemma 4.9 (Inessential Mobius Band). *Suppose B is an embedded inessential mobius band in K_I . Let H denote any one of the 1-handles contained in B . Let H^+ denote the union of H and a small neighborhood of $\partial_f K_I \cap H$ in the triangle of K containing H . There is a 2-complex K' with $K - \text{int}(H^+) \subset K'$ and a map $f' : K' - \text{int}(H^+) \rightarrow M$ which is a π_1 isomorphism. The map f' agrees with f on $K - \text{int}(H^+)$. Moreover, the I-bundle part of K'_V is $K'_I = K_I - H$ together with a single 0-handle attached along a proper geodesic edge of K_I . Furthermore, $\text{length}(\partial_f K'_V) \leq \text{length}(\partial K_V) + \text{length}(\partial B)$.*

Proof Lemmas 4.1 and 4.2 imply that if we attach discs to ∂B and a good core γ which misses H , then removal of the interior of H does not change the fundamental group. The existence of K' as stated follows from the discussion above. \square

Noting that, by Lemma 3.9 (i), K_I is composed of finitely many handles we have:

Lemma 4.10. *Suppose M is a closed hyperbolic 3-manifold, P is a triangular presentation of $\pi_1(M)$, and $\text{diam}(M) > C_3 \ell(P)$. Let V be a deep tube in M . There is a 2-complex K' and a π_1 -isomorphic map $f' : K' \rightarrow M$ constructed by surgery on K such that the subcomplex $\partial_f K'_V = f'^{-1}(V)$ has the following property. If K'_I the I-bundle part K'_V , then every properly embedded annulus and mobius band in K'_I is essential in K' and $\text{length}(\partial_f K'_V) \leq 2 \text{length}(\partial_f K_V)$.*

Proof Apply Lemma 4.5 so that every embedded annulus in K_I is essential. Next, apply Lemma 4.9 repeatedly to construct K' . For each mobius band B , the use of Lemma 4.9 adds at most $\text{length}(\partial B)$ to the length of the boundary of the complex. Notice that since Lemma 4.5 has already cut every inessential annulus in K_I , there does not exist a pair of inessential mobius bands which share a common 1-handle. For, if such a pair exists one sees an inessential annulus by removing the common 1-handle. Therefore, inessential mobius bands are disjoint. This means that if we sum the lengths of all the inessential mobius bands in K_I , Lemma 3.10 (Short Boundary) implies:

$$\text{length}(\partial_f K'_V) < \text{length}(\partial_f K_V) + \sum_{B \subset K_I} \text{length}(\partial B) < 2 \text{length}(\partial(K_V))$$

\square

The principal use of this fact is to get a bound on the first homology rank of the I-bundle portion of the 2-complex. We do this by taking stock of our various results bounding length. First, the Margulis Lemma implies that essential loops on $\partial_f K_V$ always have length at least the Margulis Constant $\tilde{\epsilon}$. Second, Lemma 4.6 tells us that we may assume every annulus and mobius band in K_I is essential. So each has length at least $\tilde{\epsilon}$. Moreover, we are guaranteed that $\text{length}(\partial_f K') \leq 2 \text{length}(\partial_f K_V) \leq 4\pi \ell(P)$. In our situation, the number of annuli and mobius bands gives a bound on $\text{Rank}(H_1(K_I))$. Thus, we have the following:

Lemma 4.11. *Let G be a finite metric graph with the following properties:*

- (i) $\text{length}(G) < N$
- (ii) *Every simple closed curve in G has length at least ϵ .*

Then $\text{Rank}(H_1(G)) \leq \frac{32N^2}{\epsilon^2}$.

Proof Let T be a maximal tree of G . We wish to bound the number of edges in $G - T$ since this bounds the first homology rank. Now, given any integer $m > 0$, condition (i) implies $G - T$ contains at most m edges of length at least $\frac{N}{m}$. So if we choose $m = \frac{5N}{\epsilon}$, $G - T$ contains at most $\frac{5N}{\epsilon}$ edges of length at least $\frac{\epsilon}{5}$. Hence, let \mathcal{L} denote the collection of edges in $G - T$ of length less than $\frac{\epsilon}{5}$. If \mathcal{L} is empty or contains only one edge, the proof is complete. Otherwise, given any pair of edges A and B in $G - T$, one vertex of A has distance at least

$$\frac{1}{2}(\epsilon - \text{length}(A) - \text{length}(B)) \geq \frac{1}{2}(\epsilon - 2\frac{\epsilon}{5}) = \frac{3\epsilon}{10}$$

from one vertex of B . This follows since otherwise we can construct a simple closed curve of length less than ϵ which contradicts condition (ii). Choose a maximal set of vertices S in T such that for every pair of vertices v and w in S , we have $d(v, w) \geq \frac{2\epsilon}{5}$. We may center a family of pairwise disjoint balls of radius $\frac{\epsilon}{5}$ at the vertices in S . Now notice that each such ball must contain an edge path of length at least $\frac{\epsilon}{5}$. Since the balls are pairwise disjoint, there are at most m such balls. Thus, none of the edges in \mathcal{L} has both endpoints in a single ball as this would also violate condition (ii). On the other hand, if two of the edges of \mathcal{L} connect the same pair of balls, there is a simple closed curve of length less than ϵ which also contradicts condition (ii). Therefore, there are at most

$$\binom{m}{2} = \frac{1}{2}m(m-1) \leq \frac{1}{2}\left(\frac{8N}{\epsilon}\right)^2$$

edges in $G - T$ which completes the proof. \square

It now follows easily that the first homology of K'_V is similarly bounded.

Theorem 4.12 (Bounded Homology). *Suppose M is a closed hyperbolic 3-manifold, P is a triangular presentation of $\pi_1(M)$, and $\text{diam}(M) > C_3\ell(P)$. Let V be a deep tube in M . There is a 2-complex K and a π_1 -isomorphic map $f : K \rightarrow M$ such that the subcomplex $\partial_f K_V = f^{-1}(V)$ has $\text{Rank}(H_1(K'_V)) \leq B_1\ell(P)^2$ where $B_1 = (\frac{128\pi^2}{\epsilon^2} + 3)$ and $\text{length}(\partial_f K_V) \leq 4\pi\ell(P)$.*

Proof Notice that K'_I can be retracted onto a spine G which is a graph with the following properties: $\text{length}(G) < \text{length}(\partial_f K'_I) < 4\pi\ell(P)$ and every simple closed curve in G has length at least $\tilde{\epsilon}$. Hence Lemma 4.11 gives that $\text{Rank}(H_1(K'_I)) \leq \frac{32(4\pi\ell(P)^2)}{\tilde{\epsilon}^2}$. Now K'_V is built by attaching at most $\ell(P)$ *monkey*-handles to K'_I . Attaching each *monkey*-handle corresponds to attaching a trivalent vertex to the spine; so we can bound the homology of K'_V by watching what happens as we attach tiny neighborhoods of trivalent vertices to the spine of K_I . It is an easy argument that each trivalent vertex addition increases the first homology rank by at most 3. Thus, the lemma is proved. \square

While the above theorem tells us that K'_V has bounded first Betti number, we have done nothing to bound the complexity of $\partial K'_V$. We have no bound on the number of 0-handles in K'_V . Indeed, we have added 0-handles during our surgery

procedure. Each 0-handle contributes one edge to $\partial_f K'_V$, so it is possible that $\partial_f K'_V$ has a large number of short edges. For our subsequent arguments, it is also necessary to bound the number of edges in $\partial_f K'_V$. Hence, we introduce a final surgery procedure to remove all but a bounded number of 0-handles. As shown in our previous discussion, a zero 0-handle in K'_V is a disc with contained in a single triangle of K' . The boundary of the 0-handle consists of two edges. One edge is a subarc of an edge of the triangle. The other edge is contained in $\partial_f K'_V$. Consider the special case in which two 0-handles are joined together by two triangles of K' glued along a common edge as shown in Figure 4.2. This gives a disc \mathbb{D} in K'_V with $\partial\mathbb{D}$ an embedded loop in $\partial_f K'_V$. As in the proof of Lemma 4.5 (Inessential Annulus), using Corollary 4.4, attach a disc to $\partial\mathbb{D}$ then remove one of the 0-handles. This operation also removes an edge of $\partial_f K'_V$. In the following, we generalize this procedure to all of K'_I . Let L be an embedded loop in $\partial_f K'_I$. We note that L has a natural subdivision into edges as follows: if H is a 1-handle or 0-handle of K'_I and $H \cap L$ is non-empty, then $\partial H \cap L = H \cap L$, so that $H \cap L$ either a single arc or a pair of disjoint arcs. The collection of these arcs taken over all 0-handles and 1-handles that meet L thus gives an edge subdivision of L . We call an edge of L that is contained in the boundary of a 0-handle a **0-handle edge**.

Proposition 4.13. *Let $f : K \rightarrow M$ be a triangular complex. Assume that $\text{diam}(M) > C_3 \ell(P)$. Let $f' : K' \rightarrow M$ be the complex given by Lemma 4.10. Let K'_I denote the complement of the interiors of the monkey-handles in K'_V . Let L be an embedded loop in $\partial_f K'_I$ with edge subdivision as above. If L contains a 0-handle edge, then L contains two 0-handle edges and L is inessential in K' .*

Proof Let \mathcal{A} denote the union of all 0-handles and 1-handles in K'_I that contain an edge of L . By hypothesis, there is at least one 0-handle H in \mathcal{A} corresponding to the 0-handle edge of L . If H is joined to another 0-handle by two triangles of K' glued along a common edge, then since L is embedded, these are the only two handles in \mathcal{A} and the proof is complete. So suppose that H is attached to a 1-handle H_1 of \mathcal{A} . Homotop L into H_1 by pushing the 0-handle edge of L through H into H_1 . Likewise, we can homotop L through H_1 into the next handle. Continuing in this manner, we must eventually reach a final 0-handle disjoint from H since L is embedded. Contract L to a point in this 0-handle to complete the proof. \square

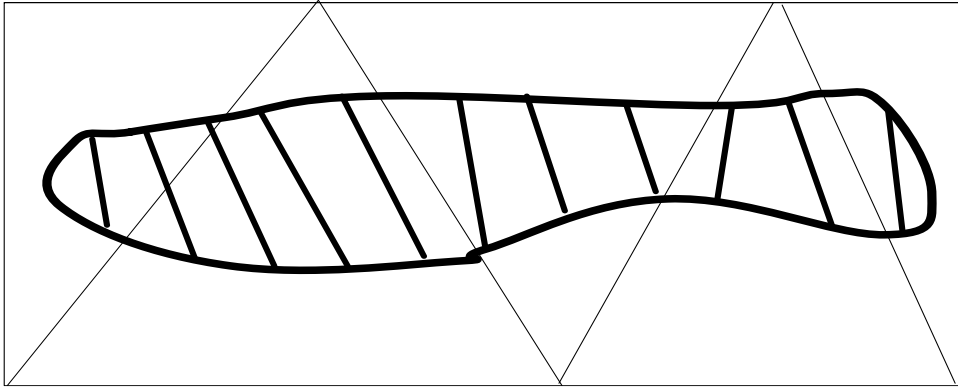


Figure 4.2.

Lemma 4.14 (0-handle surgery). *Let $f : K \rightarrow M$ be a triangular complex. Assume that $\text{diam}(M) > C_3\ell(P)$. Let $f' : K' \rightarrow M$ be the complex given by Lemma 4.10. Let K'_I denote the I-bundle portion of K'_V . Let L be an embedded loop in $\partial_f K'_I$ with edge subdivision as above. Suppose that L contains a 0-handle edge and let H denote the corresponding 0-handle. Define H^+ to be the union of H and a small neighborhood of $\partial_f K'_I \cap H$ in the triangle of K containing H . There is a 2-complex K'' with $K' - \text{int}(H^+) \subset K''$ and a map $f'' : K'' - \text{int}(H^+) \rightarrow M$ is a π_1 isomorphism. Moreover, $K''_V = K'_V - H$.*

Proof Attach a disc to L and remove the interior of H^+ . The result follows from Lemma 4.1 and Corollary 4.4. \square

By applying this result repeatedly, we have the following theorem.

Theorem 4.15 (Good Complex). *Let $f : K \rightarrow M$ be a triangular complex. Assume that $\text{diam}(M) > C_3\ell(P)$. Let $f' : K' \rightarrow M$ be the complex given by Lemma 4.10. There is a π_1 -isomorphic map of a 2-complex $f'' : K'' \rightarrow M$ constructed by surgery on K' such that, if K''_I denotes the I-bundle part of K''_V , then no simple loop in $\partial_f K''_I$ contains a 0-handle edge. Moreover, $\text{Rank}(H_1(K''_V)) \leq B_1\ell(P)^2$ and $\text{length}(\partial_f K''_V) \leq 2\text{length}(\partial_f K'_V)$.*

Henceforth, we shall refer to $f'' : K'' \rightarrow M$ as constructed above as a **good complex**. Using this definition, Theorem 4.15 (Good Complex) gives:

Corollary 4.16. *Let $f : K \rightarrow M$ be a good complex. Assume that $\text{diam}(M) > C_3\ell(P)$. The $\partial_f K_V$ contains at most one 0-handle edge for each component of K_I . Hence, there are at most $3\ell(P)$ 0-handle edges.*

Proof If a component of K_I has two attached 0-handles, then there is a loop in $\partial_f K_I$ containing two 0-handle edges. This cannot happen in a good complex. Also, there are at most three components of K_I attached to each *monkey-handle*. Since there are at most $\ell(P)$ *monkey-handles*, the result follows. \square

We have now shown that a good complex K has a bounded number of 0-handle edges in $\partial_f K_V$. Since each *monkey-handle* in K_V contributes three edges to $\partial_f K_V$, our above discussion shows that we need only bound the number of edges added to $\partial_f K_V$ by 1-handles. In fact, no such bound exists; it is possible to have a large number of 1-handles which attach very short edges to K_V . The difficulty is that $\partial_f K_V$ need not be a closed graph. Hence it may contain many univalent vertices. Instead, we show that there is a closed subgraph, denoted G_K , of $\partial_f K_V$ with the properties we need.

Lemma 4.17. *Let G be a closed graph with $\text{Rank}(H_1(G)) = R$, where $R > 1$. There is a subdivision of G into at most $3(R - 1)$ edges.*

Proof Let G have an arbitrary subdivision into edges and vertices. Since G is closed, there are no univalent vertices. Given a pair of edges that meet at a bivalent vertex, amalgamate these two edges into a single edge, thus removing the bivalent vertex. Continue in this manner until every vertex in G has valence at least three. It then follows that three times the number of vertices is at most twice the number of edges. But

$$R = 1 + (\text{number of edges}) - (\text{number of vertices})$$

whence the result follows. \square

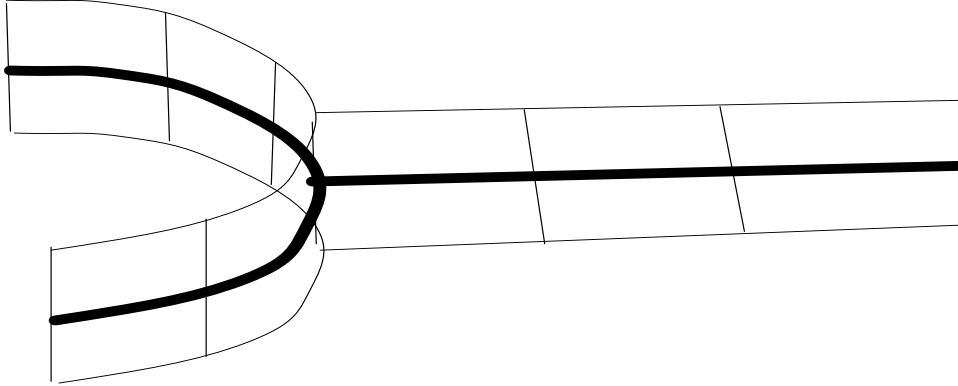


Figure 4.3. The spine G of K_1 .

Lemma 4.18. *Let $f : K \rightarrow M$ be a good complex. Assume that $\text{diam}(M) > C_3 \ell(P)$. Then $\text{Rank}(H_1(\partial_f K_V)) \leq B_2 \ell(P)^2$ where $B_2 = B_1 + 6$.*

Proof Let K_1 denote the subcomplex of K_I consisting only of 1-handles. It follows that $H_1(K_1) = H_1(K_I)$. There is a spine G of K_1 with a subdivision into edges and vertices as follows: each 1-handle joins a pair of edges of a triangle in K . Join these edges with a properly embedded arc in the 1-handle. The arc then gives an edge of G and its endpoints two vertices of G . Continue in this manner while choosing the arcs so that their endpoints agree on adjacent 1-handles. This constructs and subdivides G . Let $\partial_f K_1 = f^{-1}(f(K_1) \cap \partial V)$. Now notice that this construction gives a natural subdivision of $\partial_f K_1$ into edges and vertices such that $\partial_f K_1$ has exactly twice as many edges and exactly twice as many vertices as G . It follows at once that $H_1(\partial_f K_1) = H_1(G)$ so that in fact $H_1(\partial_f K_1) = H_1(K_I)$. Now $\partial_f K_V$ is constructed by attaching 0-handle edges and edges from *monkey*-handles to $\partial_f K_1$. Each *monkey*-handle attaches three edges and Corollary 4.16 shows there are at most $3\ell(P)$ 0-handle edges. Thus, since there are at most $\ell(P)$ *monkey*-handles, we attach at most $6\ell(P)$ edges to $\partial_f K_1$ to obtain $\partial_f K_V$. This means that

$$\text{Rank}(H_1(\partial_f K_V)) \leq \text{Rank}(H_1(G)) + 6\ell(P).$$

Now using Theorem 4.12 (Bounded Homology) and that $H_1(K_1) = H_1(K_I)$ we have

$$\text{Rank}(H_1(\partial_f K_V)) \leq \ell(P)^2 + 3\ell(P)$$

which proves the lemma. \square

We combine Lemma 4.18 and Lemma 4.17 to show that $\partial_f K_V$ has a closed subgraph with a bounded number of edges with first homology isomorphic to $H_1(\partial_f K_V)$.

Theorem 4.19 (Good Subgraph). *The graph $\partial_f K_V$ contains a closed subgraph G_K with the these properties:*

- (i) $H_1(G_K) = H_1(\partial_f K_V)$.
- (ii) G_K has a subdivision into at most $3B_2 \ell(P)^2$ edges.
- (iii) $\text{length}(G_K) \leq \text{length}(\partial_f K_V)$

Proof Choose a maximal closed subgraph G_K of $\partial_f K_V$. (i) and (iii) are obvious and (ii) follows from application of Lemma 4.17 and Lemma 4.18. \square

We call a graph G_K provided by Theorem 4.19 a **good subgraph**.

5. PROOF OF MAIN THEOREM

Suppose the $f : K \rightarrow M$ is a good complex and assume that $\text{diam}(M) > C_3 \ell(P)$, where P is some presentation of $\pi_1(M)$. We can prove some results about the homology of loops in $\partial_f K_V$. To be clear, we have an upper bound on the length of $\partial_f K_V$ which precludes (via Corollary 3.8) the existence of embedded loops which map to powers of the meridian in ∂V . On the other hand, the Lemma 4.0 shows that there must exist at least one loop in $\partial_f K_V$ which does map to an essential loop in ∂V . Hence, the image of this essential loop is also essential in M . Let $[\lambda]$ and $[\mu]$ denote longitude and meridian basis for $\pi_1(\partial V)$.

Lemma 5.0. *Suppose M is a closed, orientable hyperbolic 3-manifold, P is a triangular presentation for $\pi_1(M)$, and $f : K \rightarrow M$ is its associated good complex. If $\text{diam}(M) > C_3 \ell(P)$ then there exist a pair of embedded loops α and β in $\partial_f K_V$ such that both are essential in K and $\Delta([\alpha], [\beta]) \neq 0$.*

Proof By Lemma 4.0, there exists at least one such essential loop. Label this loop α . Since $\text{diam}(M) > C_3 \ell(P)$, we have that $[\alpha] \neq [\mu]^n$ for any integer n . Suppose that $\langle [\alpha] \rangle = f_*(H_1(\partial V))$ and let G denote the component of K_V containing α . For simplicity, assume every other component of G has inessential image in ∂V . The argument in the proof of Lemma 4.0 shows that we may assume that each of the other components of $\partial_f K_V$ maps to a single point in $\partial_f K_V$. We can generate $H_1(G)$ using α together with a collection of simple closed curves in G , none of which is homologous to α in G . The image of each simple closed curve is null homologous in $H_1(\partial V)$. Thus, we may modify f homotopically so that each of these simple closed curves is mapped to a single point. A second homotopy of f makes the image $f(\alpha)$ embedded. If other components of G have essential image in ∂V , we can repeat the above procedure on each. Since there are only finitely many components of G , we can produce a finite collection of embedded loops parallel to $f(\alpha)$ which form the image of G . We can then select another loop in ∂V parallel to $f(\alpha)$ that misses $f(K)$, contradicting Lemma 2.3. \square

Proposition 5.1. *Suppose A , B , and μ are closed geodesics on a Euclidean torus T with $\Delta(A, B) \neq 0$. Then the following inequality holds:*

$$\frac{1}{2} \frac{\ell(\mu)}{\ell(A) + \ell(B)} \leq \max\{|\Delta(\mu, A)|, |\Delta(\mu, B)|\}.$$

Proof Consider the set of lifts of A and B to the universal cover of T . These form a family of parallelograms in the plane. Each parallelogram has one pair of sides of length $\frac{\ell(A)}{|\Delta(A, B)|}$ and another pair of length $\frac{\ell(B)}{|\Delta(A, B)|}$. Notice that the diameter D of any one of these parallelograms satisfies

$$\ell(D) \leq \frac{\ell(A) + \ell(B)}{|\Delta(A, B)|}.$$

It follows that $\ell(\mu) \leq [|\Delta(\mu, A)| + |\Delta(\mu, B)|]\ell(D)$. One finds immediately that

$$\begin{aligned} \frac{\ell(\mu)}{\ell(A) + \ell(B)} &\leq \frac{\ell(\mu)}{\ell(A) + \ell(B)} |\Delta(A, B)| \leq |\Delta(\mu, A)| + |\Delta(\mu, B)| \\ &\leq 2\max(|\Delta(\mu, A)|, |\Delta(\mu, B)|). \square \end{aligned}$$

If we combine these two results with with Lemma 3.10 (Short Boundary), we find that $\partial_f K_V$ contains an embedded loop which is homologous to a huge power of the core curve of the deep tube.

Corollary 5.2. *Suppose M is a closed, orientable hyperbolic 3-manifold, P is a triangular presentation for $\pi_1(M)$, and $f : K \rightarrow M$ is an associated good complex. Let $[\gamma]$ denote the homotopy class of the core of the deep tube of M . Let $C > C_3$. If $\text{diam}(M) > C\ell(P)$, then there exists an embedded loop L in $\partial_f K_V$ such that in $\pi_1(M)$, $f_*([L]) = [\gamma]^N$ where $N \geq \frac{\sinh(C\ell(P))}{4\ell(P)}$.*

Proof Let α and β denote the two loops provided by Lemma 5.1. It follows from this lemma that $\Delta(f(\alpha), f(\beta)) \neq 0$. Also, Lemma 3.10 (Short Boundary) gives that $\ell(f(\alpha)) + \ell(f(\beta)) \leq 4\pi\ell(P)$. By applying Proposition 5.1 to geodesic representatives of $f(\alpha)$ and $f(\beta)$, we conclude at once that

$$\max\{|\Delta(\mu, f(\alpha))|, |\Delta(\mu, f(\beta))|\} \geq \frac{\ell(\mu)}{8\pi\ell(P)}.$$

Using Lemma 3.3 (Boundary Torus) and $\text{diam}(M) \geq C\ell(P)$ we obtain

$$\max\{|\Delta(\mu, f(\alpha))|, |\Delta(\mu, f(\beta))|\} \geq \frac{\sinh(C\ell(P))}{4\ell(P)}.$$

Now the algebraic intersection of a loop with the meridian equals the number of times the loop wraps around the core. This proves the statement by taking γ to be either α or β as appropriate. \square

It is helpful at this point to arrange that K_V is a connected set. This can be achieved as follows. If the loops X and β provided by Lemma 5.0 are contained in the same component, we agree to think of this component as K_V . If they are in different components, let x denote a point of ∂V where $f(X)$ and $f(\beta)$ intersect. Choose points y_1 and y_2 on X and β respectively which map to x . Join these points by an arc gluing the endpoints of an arc to y_1 and y_2 so that the arc maps to $f(X)$. The resulting complex is then connected but retains the same first homology as the original complex. In making this addition, we at most double the length of $\partial_f K_V$ so that we have $\text{length}(\partial_f K_V) \leq 8\pi e^2 \ell(P)$.

We now wish to utilize some interesting properties which are forced upon K_V by the geometry in our situation. In the following, we prove that we may assume that the map $f : K_V \rightarrow V$ induces an epimorphism between first homology groups. This is done by showing that the map f lifts to a cover of V of index bounded by $\ell(P)$.

Proposition 5.3. *Suppose V is a solid torus with Euclidean boundary T . Suppose also that $\text{Area}(T) = A$ and $\text{inj}(T) = \epsilon$. Let α and β denote a pair of loops on T*

such that $\Delta([\alpha], [\beta]) \neq 0$ and $\ell(\alpha) + \ell(\beta) < R$. There is a covering \tilde{V} of index at most $\frac{8R^2}{3\epsilon^2}$ such that the group $\langle [\tilde{\alpha}], [\tilde{\beta}] \rangle$ generates $\pi_1(\tilde{V})$.

Proof Fix a short basis $\{[X], [Y]\}$ for $\pi_1(T)$. By Proposition 3.7, we may write $[\alpha] = a[X] + b[Y]$ and $[\beta] = c[X] + d[Y]$ with $\max\{a, b, c, d\} \leq \frac{2R}{\sqrt{3\epsilon}}$. Let $[\gamma]$ denote the core class of V on T . It follows that $(ad - bc)[\gamma]$ is a finite linear combination of $[\alpha]$ and $[\beta]$. This means $\langle [\tilde{\alpha}], [\tilde{\beta}] \rangle$ is isomorphic to $n\mathbb{Z}$ where $n | (ad - bc)$. Thus, by taking an at most $|ad - bc|$ -fold cover of V we obtain the desired lift. Moreover, $|ad - bc| < 2(\frac{4R^2}{3\epsilon^2})$. \square

Corollary 5.4. *Suppose M is a closed, orientable hyperbolic 3-manifold, P is a triangular presentation for $\pi_1(M)$, and $f : K \rightarrow M$ is its associated good complex. If $\text{diam}(M) > C_3\ell(P)$, there is a covering $\rho : \mathcal{Y} \rightarrow V$ of index at most $\frac{32(\pi\ell(P))^2}{3\epsilon^2}$ and a lift $\tilde{f} : K_V \rightarrow \mathcal{Y}$ such that $\tilde{f}_* : \pi_1(K_V) \rightarrow \pi_1(\mathcal{Y})$ is an epimorphism.*

Proof Apply Proposition 4.3 using the images under f of the loops α and β in $\partial_f K_V$ as provided by Lemma 5.1. \square

If we assume that $\text{diam}(M) > C_3\ell(P)$, we have in effect constructed a 2-complex $f : K_V \rightarrow V$ which has very strange properties. First, Theorem 4.12 tells us that the rank of $H_1(K_V)$ is bounded by a constant times presentation length. Second, we know that the induced homomorphism on first homology is surjective. Third, by Corollary 5.4, we know that $\partial_f K_V$ contains a short loop which maps to a huge power of the core of V . We will now modify K_V to show that this leads to a contradiction. The first step is to show that we can attach a small number of discs to K_V to obtain a new complex which almost gives an H_1 -isomorphism. This follows by using a good subgraph of $\partial_f K_V$.

Corollary 5.5. *Suppose M is a closed, orientable hyperbolic 3-manifold, P is a triangular presentation for $\pi_1(M)$, and $f : K \rightarrow M$ is its associated good complex. Let G_K be any good subgraph of $\partial_f K_V$. If $\text{diam}(M) > C_3\ell(P)$, then the map $\tilde{f} : G_K \rightarrow \mathcal{Y}$ induces an epimorphism of first homology groups.*

Consider a good subgraph G_K . Notice that there is a collection of simple closed curves d_1, \dots, d_n such that $\{[d_1], \dots, [d_n]\}$ is a basis for $H_1(G_K)$. Now by Lemma 4.19 (Good Subgraph), the length of each loop d_i is less than $\text{length}(G_K) \leq 4\pi(\ell(P))$. The Lemma 4.19 also tells us there are at most $B_2\ell(P)^2$ such loops since this is the upper bound on the first homology rank of $H_1(G_K)$. Therefore, the sum of the lengths of these loops is bounded above by the product of these two quantities:

$$\sum_{i=1}^n \ell(d_i) \leq B_2\ell(P)^2 \times 4\pi(\ell(P)).$$

We call such a basis for $H_1(G_K)$ a **small basis**. G_K is mapped into a torus with bounded geometry via an H_1 -epimorphism. We use the idea of a small basis to show that there are restrictions on the complexity of loops which represent generators of the kernel of the induced first homology map.

Lemma 5.6. *Suppose that G is finite metric graph with $\text{Rank}(H_1(G)) \leq B$ and $\text{length}(G) \leq L$.*

Suppose W is a solid torus with $T = \partial W$ a Euclidean torus with $\text{Area}(T) =$

A and $\text{inj}(T) = R$. Given a continuous map $g : G \rightarrow W$ such that g_* is an epimorphism of first homology groups, there are loops s_1, \dots, s_m in G , where $m = \text{rank}(H_1(G)) - 2$ such that $\{[s_1], \dots, [s_m]\}$ is a basis for the kernel of $g_* : H_1(G) \rightarrow H_1(T)$. Furthermore, for each $1 \leq i \leq m$, $[s_i] = c_{i1}[d_1] + \dots + c_{in}[d_n]$ where $[d_1] \dots [d_n]$ is a small basis for $H_1(G)$ and $|c_{ij}| \leq (\frac{8B^2L^2}{3R^2})$.

Proof We have a homomorphism $g_* : H_1(G) \rightarrow H_1(W)$. We can write g_* as a matrix with integer coefficients

$$g_* = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix}$$

using the small basis $\{[d_1], \dots, [d_n]\}$. Since $\sum_{i=1}^n \ell(d_i) \leq BL$ we have at once that $\ell(d_i) \leq BL$ for all $i = 1 \dots n$. Thus, we can use Proposition 3.7 to give an upper bound on the $|a_{ij}|$. That is, using a short basis for $\pi_1(T)$, we conclude that $|a_{ij}| \leq \frac{2BL}{\sqrt{(3)R}}$ for all $1 \leq i, j \leq n$.

Now, since g_* is an epimorphism, this matrix extends to an onto map from \mathbb{R}^m to \mathbb{R}^2 . Hence, it is not the case that every 2×2 submatrix of g_* has determinant zero. This means there is a basis for the kernel of g_* using vectors of the form

$$(c_{i1} \quad c_{i2} \quad 0 \quad \dots \quad c_{ij} \quad 0 \quad \dots \quad 0)$$

where the entries c_{ij} correspond to those in the cross product of

$$(a_{11} \quad a_{12} \quad a_{ij})$$

with

$$(a_{21} \quad a_{22} \quad a_{2j})$$

It follows that the coefficients c_{ij} are integers that are bounded so that for all $1 \leq i, j \leq n$,

$$|c_{ij}| \leq 2 \max_{ij} (|a_{ij}|)^2 \leq 2 \left(\frac{2BL}{\sqrt{(3)R}} \right)^2.$$

□

Applying this result to G_K and \mathcal{Y} as above, in our situation we have:

Corollary 5.7. *There are loops s_1, \dots, s_m in G_K such that $\{[s_1], \dots, [s_m]\}$ is a basis the kernel of $f_* : H_1(G_K) \rightarrow H_1(\mathcal{Y})$. Furthermore, for each $1 \leq i \leq m$, $[s_i] = c_{i1}[d_1] + \dots + c_{in}[d_n]$, we have $|c_{ij}| \leq B_3 \ell(P)^6$ where $B_3 = \frac{512\pi^2 B_2^2}{3\tilde{\epsilon}^2}$*

Proof Apply Lemma 5.6 with $B = B_2 \ell(P)^2$, $L = 8\pi \ell(P)$ and $R = \tilde{\epsilon}$. □

Finally, we shall need the following algebraic bound in the proof of the main theorem below:

Proposition 5.8. *Suppose P is a presentation of \mathbb{Z}_N . Then $\ell(P) \geq N^{\frac{1}{\sqrt{\ln(N)}}} + \sqrt{\ln(N)} - 1$.*

Proof Fix $N > 0$, let P be a presentation of \mathbb{Z}_N . Since P is a presentation of an Abelian group, we have an integer presentation matrix $A = (a_{ij})$ for P . We can then define $\ell(P) = \sum |a_{ij}|$. We may assume that A is a $k \times k$ matrix where $k \leq N$. To see this, note that if A is not square, we can use column and row operations to

produce a block presentation matrix B which contains a $q \times q$ presentation matrix in its upper right corner and zeros everywhere else. If B contains more columns than rows, then at least one linear combination of the generators of P is infinite cyclic, which is clearly impossible since P presents \mathbb{Z}_N . Likewise, if B contains more rows than columns, one or more of the rows are linear combinations of the others. Hence, we may choose a maximal collection of linearly independent rows. By throwing away the remaining rows, we find the resulting matrix presents a group \mathbb{Z}_S with $S > N$. Notice that if A is square and $k > N$, then $\sum |a_{ij}| > N$, so that the 1×1 presentation (N) has shorter length than that given by A . All of this means we may assume that A is a nonsingular square matrix. In fact, this implies that $\det(A) = N$. Now it is well known that

$$|\det(A)| \leq \prod \|A_i\|_1$$

where $\|A_i\|_1$ denotes the L^1 norm of the i th row of A . An easy lower bound for the norm is:

$$\|A\|_1 \geq N^{\frac{1}{k}} + k - 1$$

for every $k \times k$ matrix that presents \mathbb{Z}_N . This follows since if $\|A_i\|_1 < N^{\frac{1}{k}}$ for all $1 \leq i \leq k$, then the above inequality gives that $|\det(A)| < N$. Since this is not the case, there is at least one element a_{ij} of A with $|a_{ij}| \geq N^{\frac{1}{k}}$. The bound follows since the $k - 1$ remaining rows must each contain an entry of absolute value at least 1.

Fix a value of N . Consider the function

$$h(k) = N^{\frac{1}{k}} + k - 1.$$

If we minimize this function on the interval $[1, N]$, we find the absolute minimum is bounded below by $k = \sqrt{\ln(N)}$ so that

$$h(k) \geq N^{\frac{1}{\sqrt{\ln(N)}}} + \sqrt{\ln(N)} - 1$$

gives a lower bound for every presentation of \mathbb{Z}_N . \square

We can now prove the following:

Theorem 5.9. *There is an explicit constant $R > 0$ such that if M is a closed, connected, hyperbolic 3-manifold, and P is any presentation of its fundamental group, then $\text{diam}(M) < R(\ell(P))$.*

Proof of Main Theorem We argue by contradiction. We show that it is possible to choose $R > C_3$ sufficiently large so that $\text{diam}(M) > R\ell(P)$ is impossible. We shall construct the 2-complex K_V^+ by attaching 2-cells to the loops $s_i = c_{i1}d_1 + \cdots + c_{in}d_n$ with the coefficients c_{ij} provided by Corollary 5.7. The proof of Theorem 4.12 (Bounded Homology) and our discussions above show that we may perform this construction so that the resulting complex can be triangulated with a bounded number of triangles. To see this, using the argument in Theorem 4.12 (Bounded Homology), retract K_V onto its spine G . In doing this, every 0-handle collapses to a vertex of G . This graph has the same properties as the spine of K_I , except that we must add a small neighborhood of a trivalent vertex for each monkey handle. Let T denote a maximal tree of G . Crush this tree to a point. This produces a

π_1 -isomorphic spine G' . Moreover, it follows that from Theorem 4.12 (Bounded Homology) that G' has at most $B_1\ell(P)^2$ edges. Let K_V+ denote the complex built by gluing discs around the loops which correspond to $s_i = c_{i1}d_1 + \cdots + c_{in}d_n$ as provided by Corollary 5.7. Hence we can triangulate by coning each of the attached 2-cells. The i th 2-cell then contains at most

$$(c_{i1} + \cdots + c_{in}) \times B_1\ell(P)^2 \leq B_1\ell(P)^2 \times B_3\ell(P)^6$$

triangles. Since the number of attached 2-cells is less than the rank of $H_1(K_V)$, we obtain from this result and Theorem 4.12 (Bounded Homology)

that K_V^+ can be triangulated with at most

$$B_1^2 B_3 \ell(P)^8$$

triangles.

There is an obvious continuous map $f : K_V^+ \rightarrow Y$ which induces an epimorphism of first homology groups. There are essentially two possibilities for $H_1(K_V^+)$:

Case (1) $H_1(K_V^+) \cong \mathbb{Z} \oplus \mathbb{Z}$. We prove this cannot occur:

We may form the space K_V^{++} by attaching enough discs to K_V^+ to make $\pi_1(K_V^{++}) \cong \mathbb{Z} \oplus \mathbb{Z}$. We now consider our original good complex $f : K \rightarrow M$. We can view K_V^{++} as a subset of K . To see this, notice that we constructed K_V^{++} by attaching discs to loops that were inessential in K . Thus, by Lemma 4.2 we may assume K_V^{++} is contained in K . Then the restrictions of the map f induce a pair of homomorphisms $i_1 : \pi_1(\partial_f K_V^{++}) \rightarrow \pi_1(V)$ and $i_2 : \pi_1(\partial_f K_V^{++}) \rightarrow \pi_1(M - \text{int}(V))$. The first map i_1 is injective by our construction. On the other hand, the incompressibility of ∂V in $M - \text{int}(V)$ ensures that i_2 is also injective. This is impossible since Van Kampen's Theorem then gives an injection of $\pi_1(K_V^{++}) \cong \mathbb{Z} \oplus \mathbb{Z}$ into $\pi_1(M)$ which contradicts that M is closed hyperbolic.

Case (2) $H_1(K_V^+) \cong \mathbb{Z}$ or $H_1(K_V^+) \cong \mathbb{Z} \oplus \mathbb{Z}_r$. Suppose $H_1(K_V^+) \cong \mathbb{Z}$. We show that these cases lead to a contradiction:

Attach a disc to the loop provided by Corollary 5.2. This creates the space K_V^++ which has these two properties:

- (i) $H_1(K_V^++) = \mathbb{Z}_N$ where $N \geq \frac{\sinh(R\ell(P))}{4\ell(P)}$
- (ii) K_V^++ has triangulation by $B_4\ell(P)^6$ triangles, where

$$B_4 = B_2 + B_3$$

This constant follows since the loop provided by Corollary 5.2 is embedded and there are at most B_2 edges in ∂G_K . The triangulation gives a presentation of \mathbb{Z}_N of length at most $3B_4\ell(P)^6$. By substituting the value for N in (i) into the lower bound expression from Proposition 5.6, we note that the minimum size of the triangulation in this case is a function of exponential growth with respect to R and $\ell(P)$ while the number of triangles exhibits polynomial growth. It then follows that if $R > C_3$ is sufficiently large, the construction of this complex gives a presentation for \mathbb{Z}_N which is smaller than the lower bound provided by Proposition 5.6. A similar argument works if $H_1(K_V^+) \cong \mathbb{Z} \oplus \mathbb{Z}_r$. \square

REFERENCES

- [A] Adams, Colin C., *The Noncompact Hyperbolic 3-Manifold of Minimal Volume*, Proc. A.M.S. **100** (1987) 601-606.
- [BGS] Ballman, W., Gromov, M., and Schroeder, V., *Manifolds of Nonpositive Curvature*, Birkhauser (1981).
- [BP] Benedetti, R. and Petronio, C., *Lectures on Hyperbolic Geometry*, Springer-Verlag (1992).
- [C] Cooper, Daryl, *The Volume of a Closed Hyperbolic 3-Manifold is Bounded by π Times the Length of any Presentation of its Fundamental Group*, Proc. A.M.S.
- [Th] Thurston, W.P., *The Geometry and Topology of 3-Manifolds*, Princeton University (1979).
- [W] White, M.E., *Injectivity Radius and Fundamental Group of Hyperbolic 3-manifolds*, To Appear: Communications in Analysis and Geometry.